I. Introduction to Handlebody Decompositions

Our main goal is to understand manifolds using a simple "decomposition" of them in to "handles"

it turns out all smooth manifolds have such decompositions and we can use them to understand manifolds, in all dimensions, fairly explicitly

we will
1) prove the h-cobordism theorem, and hence, the high dimensional Poincaré conjecture
2) classify surfaces using handlebodies
3) study 3-manifold where handlebody decompositions are called Heegaard decompositions

   we will see how to understand knots and surgery from this perspective
4) Study 4-manifolds using "Kirby pictures" where we can <u>see</u> things about 4-manifolds
5) hopefully many other things!

we will study manifolds with boundary and we want some "control" on the boundary such manifolds are called cobordisms

A. Cobordisms

The main objects we will study are cobordisms

a <u>cobordism</u> W^{n+1} is an oriented $(n+1)$ dimensional manifold and a splitting of its boundary

$$\partial W = (-\partial_- W) \cup (\partial_+ W)$$

reverse orientation

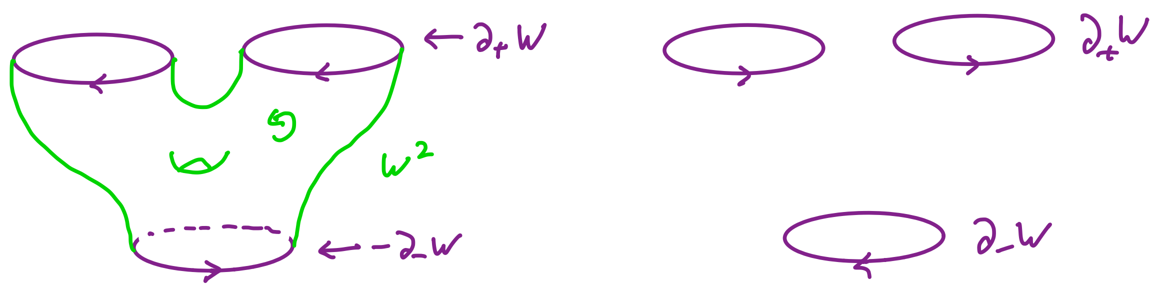
into disjoint pieces

lower boundary

upper boundary

we say W is a <u>cobordism from $\partial_- W$ to $\partial_+ W$</u>

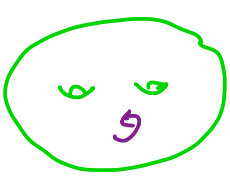
we will always assume W is <u>compact</u> (this is a standard assumption)

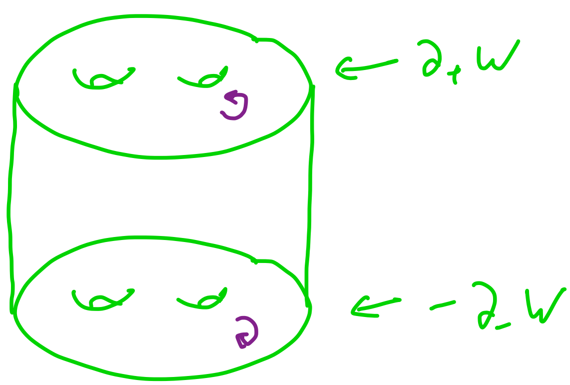


<u>denote:</u> $\partial_- W \xrightarrow{W} \partial_+ W$

<u>simple example:</u> if M^n is any compact oriented n-manifold then $W = [0,1] \times M^n$ is a cobordism from M to M

order important!





$\partial_+ W = M$

$\partial_- W = M$

recall orient ∂W by <u>outward normal first</u>

Remarks: 1) a cobordism with $\partial = \emptyset$ is just a manifold

2) a cobordism with $\partial_- = \emptyset$ is just a manifold with boundary

3) can consider cobordism without orientation (then just ignor the $-$ before $\partial_- W$)

4) note W is also a cobordism from $-\partial_+ W$ to $-\partial_- W$

we say this is W upside down and denote it $\overline{W}$

and $-W$ is a cobordism from $-\partial_- W$ to $-\partial_+ W$ and from $\partial_+ W$ to $\partial_- W$

exercise:

1) Cobordism defines an equivalence relation on the set of closed (oriented) k-manifolds

let $\Omega_k = \{\text{closed, oriented } k\text{-manifolds}\} / \sim_{\text{oriented cobordism}}$

sometimes written Ω_k^{SO}

$\Omega_k^{uo} = \{\text{closed}, k\text{-manifolds}\} / \sim$ cobordism (not nec. orient)

sometimes written $\mathcal{N}_k$

(lots of others eg. Ω_k^{spin} = closed, spin k-manifolds upto spin cobordism)

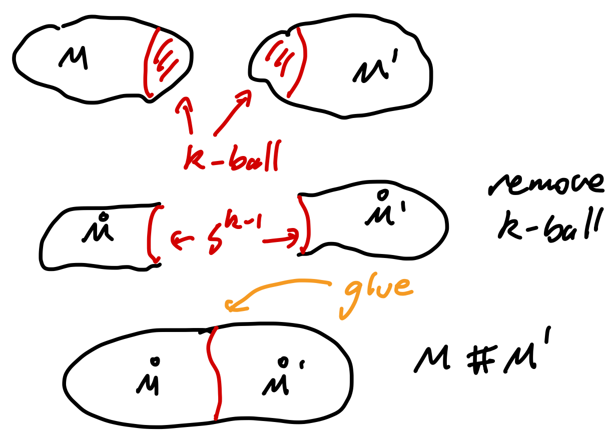
don't worry if you don't know what this is, just ignor for now

2) if M, M' are closed k-manifolds then

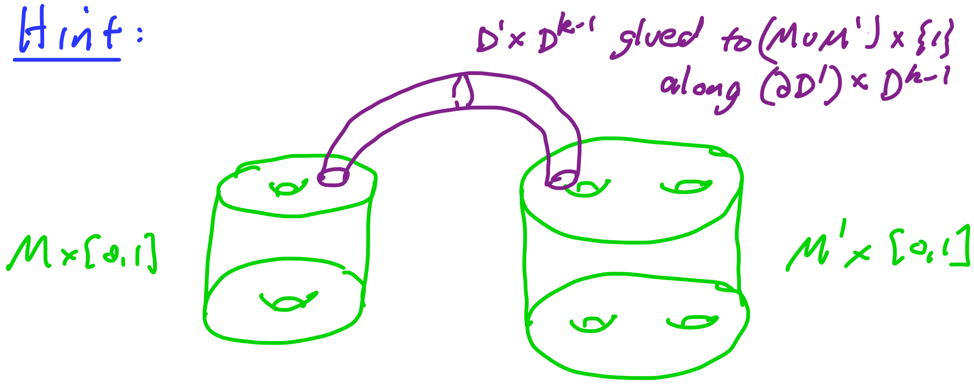
$M \amalg M'$ is cobordant to $M \# M'$

connected sum

so any $[M] \in \Omega_n$ can be realized by a connected manifold



Hint:



3) Show Ω_k (and Ω_k^{un}, Ω_k^{spin}) are groups under the operation of disjoint union (or connected sum)

- what is identity? ($\emptyset$ or any M^k that is ∂ of $(k+1)$ mfd)
- what are inverses ? (Think about $(M - B^k) \times [0,1]$)

Facts:

k	0	1	2	3	4	5
Ω_k	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$
Ω_k^{un}	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$
Ω_k^{spin}	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0

note: this says any oriented k-manifold bounds a $(k+1)$-manifold for $k=1,2,3$ but the same is not true for $k=4$

we will probably prove some of these

exercise:

1) compute Ω_k for $k=0,1,2$

(you need the classification compact of 0,1,2 manifolds for this)

2) do the same for Ω_k^{uo}

Hint: for Ω_2^{uo} try to show

$$\chi(\partial W^k) = (1-(-1)^k)\,\chi(\partial W)$$

so $\chi(\Sigma)$ mod 2 is an invariant of cobordisms

now $\chi(\mathbb{R}P^{2n})$ is odd so

$[\mathbb{R}P^{2n}] \neq 0$ in Ω_{2n}^{uo}

we also consider generalizations to cobordisms of manifolds with boundary

a cobordism of manifolds with boundary is a compact oriented manifold (with corners) W^{n+1} whose boundary is written

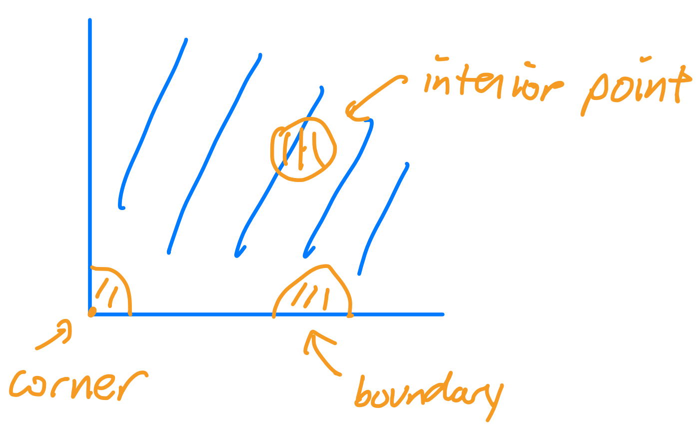
$$\partial W = (-\partial_- W) \cup (-\partial_v W) \cup (\partial_+ W)$$

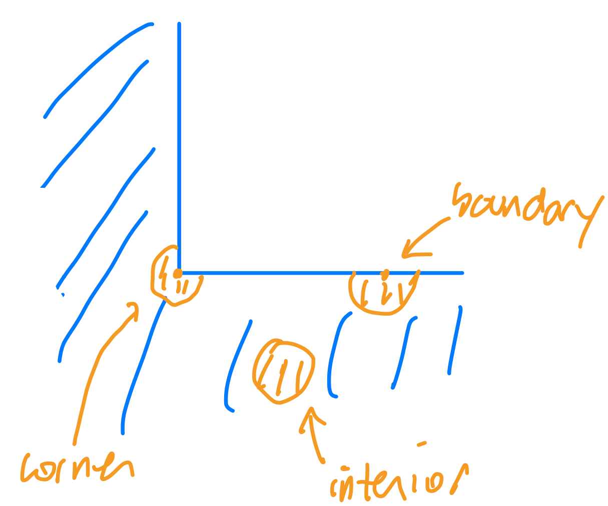
where

1) $\partial_- W \cap \partial_+ W = \emptyset$
2) $\partial_v W$ is a cobordism from $\partial_-(\partial_v W)$ to $\partial_+(\partial_v W)$
3) $\partial(\partial_+ W) = \partial_+(\partial_v W)$ and $\partial(\partial_- W) = \partial_-(\partial_v W)$

a manifold with corners is a smooth manifold with coordinate charts modeled on

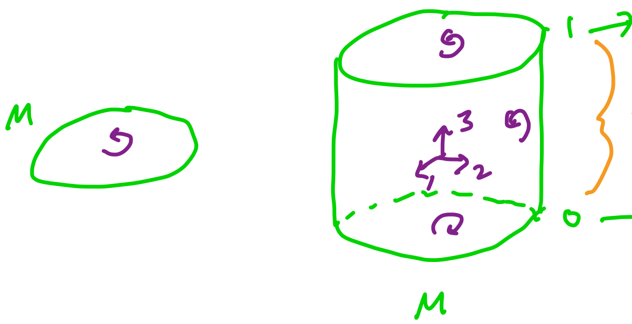
$$\{(x_1, \dots, x_n) \mid x_n, x_{n-1} \geq 0\} \qquad \{(x_1, \dots, x_n) \mid x_n \leq 0, x_{n-1} \leq 0\}$$





example:

1) let M be an oriented manifold with boundary
then $W = [0,1] \times M$ is a cobordism of
manifolds with boundary



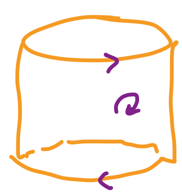
$\partial_+ W = \{1\} \times M$

$-\partial_v W = S^1 \times [0,1]$

$-\partial_- W = -\{0\} \times M$

so $\partial_\pm W = M$

$\partial_v W = S^1 \times [0,1]$

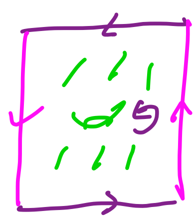


$\rightarrow$

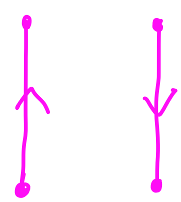
$\partial_+(\partial_v W) = \quad = \partial(\partial_+ W)$

$\partial_-(\partial_v W) = \quad = \partial(\partial_- W)$

2)



$\partial_v W =$

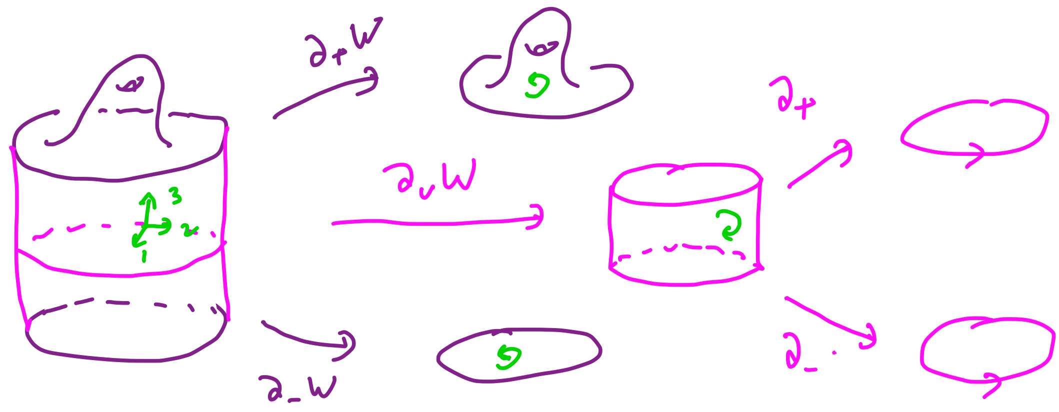


cobordism $\overset{+}{\bullet} \;\; \overset{-}{\bullet}$ to $\overset{+}{\bullet} \;\; \overset{-}{\bullet}$

$\partial_- \partial_v W \qquad \partial_+ \partial_v W$

$\partial_+ W = \bullet\!\longleftarrow\!\bullet \qquad \partial(\partial_+ W) = \overset{+}{\bullet} \;\; \overset{-}{\bullet}$

$\partial_- W = \bullet\!\longleftarrow\!\bullet \qquad \partial(\partial_- W) = \overset{+}{\bullet} \;\; \overset{-}{\bullet}$

3) 

Remark: if every component of $\partial_+ W$ and $\partial_- W$ have boundary and $\partial_v W$ is a product (i.e. $\partial_v W = [0,1] \times \partial(\partial_- W)$) then W is sometimes called a sutured manifold
(sometimes allow certain closed components e.g. Gabai's sutured 3-manifolds are allowed to have T^2 components in $\partial_v W$)

a first step to understanding cobordisms is to consider Morse functions.

B. Morse functions

a Morse function on a manifold M is a function

$$f: M \to \mathbb{R}$$

such that all critical points are non-degenerate
(if M has boundary we require all critical points to be on the interior of M)
not strictly necessary but common

recall a critical point of f is a point $p \in M$ such that

$$df_p = 0 \qquad (df_p : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R})$$

we say p is non-degenerate if the Hessian of f at p

$$Hf_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto w \cdot (v \cdot f)$$

is a non-degenerate 2-form (meaning that for a fixed $v \in T_pM$, if $Hf_p(v,w) = 0 \ \forall w \in T_pM$ then $v = 0$ and similarly if $Hf_p(v,w) = 0 \ \forall v \in T_pM$ then $w = 0$)

recall that we can think of a vector $v \in T_pM$ as a derivation

$$v : C^\infty(M) \to \mathbb{R} : f \mapsto v \cdot f$$

and $v \cdot f = df(v)$

so $v \cdot f \in \mathbb{R}$ $\therefore$ $w \cdot (v \cdot f)$ does not make sence!

to fix this we extend v to a vector field $\tilde{v} \in \mathfrak{X}(M)$

now $\tilde{v} \cdot f \in C^\infty(M)$ and $w \cdot (\tilde{v} \cdot f)$ does make sence

BUT we need to see it is independent of the extention of v to $\tilde{v}$

warning: this is not true unless p is a critical point!

to see $w.(\tilde{v}.f)$ is independent of the extension we extend w to $\tilde{w}$

now

$$[\tilde{v}.(\tilde{w}.f)](p) - [\tilde{w}.(\tilde{v}.f)](p) \overset{\text{def}^{\underline{n}}}{=} \underbrace{[\tilde{v},\tilde{w}]_p}_{\text{Lie bracket at } p} \cdot f$$

$$= df_p([\tilde{v},\tilde{w}]) = 0 \quad \leftarrow p \text{ critical pt}$$

so

$$v.(\tilde{w}.f) = [\tilde{v}.(\tilde{w}.f)](p) = [\tilde{w}.(\tilde{v}.f)](p)$$

$$= w.(\tilde{v}.f)$$

that is $w.(\tilde{v}.f)$ only depends on $v = \tilde{v}(p)$ not on the extension

also this shows

$$\boxed{Hf_p \text{ is symmetric}}$$

<u>exercise</u>:

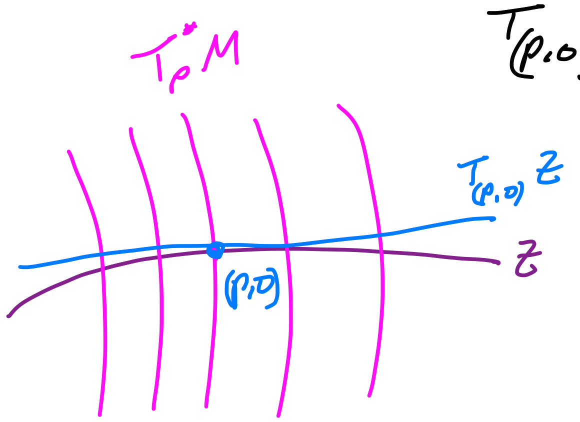
1) In local coordinates $(x_1, \dots, x_n)$ at p show that Hf_p is given by the matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right)_{i,j=1}^{n}$$

in the basis $\frac{\partial}{\partial x_1} \dots \frac{\partial}{\partial x_n}$ for T_pM

2) show the expression in 1) is independent of local coordinates (if p a crit. point)

3) note a point $(p,0) \in T^*M$ we have



$$T_{(p,0)}(T^*M) = T_{(p,0)}Z \oplus \underbrace{\ker(d\pi_{(p,0)})}_{\substack{\cong \\ T_p^*M}}$$

where $\pi : T^*M \to M$ is projection

and $Z = \text{image } \sigma_0$

$$\sigma_0 : M \to T^*M \quad \text{zero section}$$
$$p \mapsto (p,0)$$

so we have a well-defined map

$$c_p : T_{(p,0)}(T^*M) \to T_p^*M$$

only defined along zero section!!

if p is a critical point of f then define

$$(d^2f)_p : T_pW \times T_pW \to \mathbb{R}$$
$$(v,w) \mapsto c_p(d_p(df)(v))(w)$$

$df : M \to T^*M$

$d_p(df) : T_pM \to T_{(p,0)}T^*M$

$c_p(d_p(df)(v)) \in T_p^*M$

show $(d^2f)_p = Hf_p$

<u>note</u>: whenever c_p defined you get a

Hessian of f. Can use a "connection" to do this in general

4) Show p is a non-degenerate critical point of $f: M \to \mathbb{R} \Leftrightarrow df: M \to T^*M$ is transverse to the zero section Z at p

5) show non-degenerate critical points are isolated (think about what transversality means)

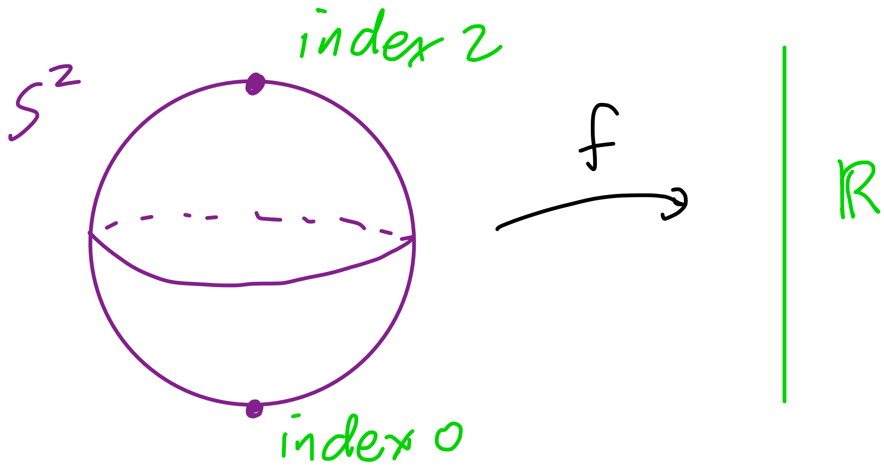
Since Hf_p is a symmetric, bilinear form we may choose a basis so that Hf_p is represented by a diagonal matrix with entries on the diagonal ± 1

we define the <u>index of p</u> to be the number of -1's

<u>examples:</u>

1) $M = S^2$ unit sphere in $\mathbb{R}^3$

$f: S^2 \to \mathbb{R} : (x, y, z) \mapsto z$



we have the standard coordinate chart

$$\phi_u : D^2 \to S^2 : (x,y) \longmapsto (x, y, \sqrt{1-x^2-y^2})$$

and

$$\phi_\ell : D^2 \to S^2 : (x,y) \longmapsto (x, y, -\sqrt{1-x^2-y^2})$$

note: $f \circ \phi_u (x,y) = \sqrt{1-x^2-y^2}$

critical point at $(0,0)$

so $(0,0,1)$ crit. pt. of f

so $d(f \circ \phi_u) = \dfrac{1}{\sqrt{1-x^2-y^2}} \begin{pmatrix} -x \\ -y \end{pmatrix}$

$$\left(\frac{\partial^2 f \circ \phi_u}{\partial x_i \partial x_j}\right) = \frac{1}{(1-x^2-y^2)^{3/2}} \begin{pmatrix} y^2-1 & -xy \\ -xy & x^2-1 \end{pmatrix}$$

so at $(0,0)$ we get $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$\therefore$ index of $(0,0,1)$ is 2

similarly the critical point $(0,0,-1)$ has index 0

exercise: show no critical points on equator.

2) recall $\mathbb{R}P^2 = \mathbb{R}^3 - \{(0,0,0)\} / \sim$ where

$(x,y,z) \sim (tx, ty, tz)$ for $t \neq 0$

$\tilde{f}(x,y,z) = \dfrac{x^2 - 2y^2}{x^2+y^2+z^2}$ induces a function on $\mathbb{RP}^2$

exercise: Show f has 3 critical points of index $0,1,2$

one may apply the Thom jet transversality theorem to establish

Thm:

> let M be any manifold
> then the set of Morse functions is residual in $C^\infty(M;\mathbb{R})$.
> Moreover, if C is a closed set on which $f: M \to \mathbb{R}$ is already Morse, then $\exists$ a Morse function $\tilde{f}: M \to \mathbb{R}$ that agrees with f on an open set containing C.

a residual set is a countable intersection of dense sets and, by definition, a Baire space is a space where residual sets are dense

$\boxed{C^\infty(M, \mathbb{R}) \text{ is a Baire space}}$

so Morse functions are dense in $C^\infty(M,\mathbb{R})$

there are simpler ways to show Morse functions exist on any manifold, but they do not give the details in the above thm and we will need these details

(might prove this later)

example 1:

If M is a compact manifold with boundary then there is a Morse function

$$f: M \to \mathbb{R}$$

such that

1) $f(M) \subset [0,\infty)$
2) 0 is a regular value of f
3) $f^{-1}(0) = \partial M$

let's see how to find such an f:

It is a standard fact from differential topology that $\exists$ an embedding

$$e: ([0,\varepsilon) \times \partial M) \to M$$

such that

- $e|_{\{0\}\times\partial M} : \partial M \to \partial M$ is the identity, and
- $im(e)$ is a neighborhood of ∂M in M

define $\tilde{f}$ on $im(e([0,\varepsilon/2]\times\partial M))$

$$\pi \circ e^{-1}: e([0,\varepsilon/2]) \to [0,\varepsilon/2]$$

where $\pi: [0,\varepsilon) \times \partial M \to [0,\varepsilon)$ is projection

and extend to be $\varepsilon/2$ on the rest of M

we can appoximate $\tilde{f}$ by a smooth function $\hat{f}$ such that

$$|\tilde{f}(x) - \hat{f}(x)| < \varepsilon/4,$$

why can we do this?

$\hat{f} = \tilde{f}$ on $e([0,\varepsilon/8] \times \partial M)$

and $\hat{f}(x) > \varepsilon/8$ for $x \in \partial M - e([0,\varepsilon/8] \times \partial M)$

note $\hat{f}$ has no critical points in $e([0,\varepsilon/8] \times \partial M)$ and 0 is a regular value

now by Th$\underline{\text{m}}$ $\exists$ a Morse function

$$f : M \to \mathbb{R}$$

such that

- $f = \hat{f}$ on a nbhd of $e([0,\varepsilon/8] \times \partial M)$ and
- $|f(x) - \hat{f}(x)| < \varepsilon/16$

so $f(x) > \frac{\varepsilon}{16}$ for $x \in e([0,\varepsilon/16) \times \partial M)$

thus $f^{-1}(0) = \hat{f}^{-1}(0) = \tilde{f}^{-1}(0) = \partial M$

0 is a regular value

$f(x) \geq 0$ for all $x \in M$

__example 2:__

a cobordism W admits a Morse function

$$f : W \to \mathbb{R}$$

such that

- $f(W) \subset [0,1]$
- $0,1$ are regular values of f
- $f^{-1}(0) = \partial_- W$ and $f^{-1}(1) = \partial_+ W$

exercise: Prove this (very similar to example 1)

example 3:

a cobordism W of manifolds with boundary where $\partial_v \cong [0,1] \times \partial(\partial_- W)$ admits a Morse function

$$f : W \to \mathbb{R}$$

such that

- $f(W) \subset [0,1]$
- $0,1$ are regular values of f
- $f^{-1}(0) = \partial_- W$ and $f^{-1}(1) = \partial_+ W$
- $f|_{\partial_v W} : ([0,1] \times \partial(\partial_- W)) \to [0,1]$ is projection

exercise: Prove this (similar to exercise 2 but need care near $\partial(\partial_- W)$ & $\partial(\partial_+ W)$)

example 4:

if Σ is a submanifold of M, then $\exists$ a Morse function

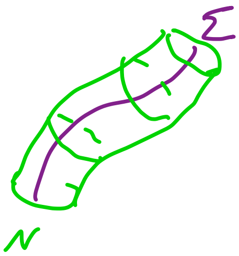
$$f : M \to \mathbb{R}$$

such that $f|_\Sigma$ is Morse, the critical points of $f|_\Sigma$ are critical points of f, and they have the same index

Proof: let $\tilde{f}: \Sigma \to \mathbb{R}$ be any Morse function

let $N \xrightarrow{\pi} \Sigma$ be a tubular nbhd of Σ in M

(i.e. N a nbhd of Σ in M and is a disk bundle over Σ)



extend $\tilde{f}$ to $\hat{f}: N \to \mathbb{R}$ by

$$\hat{f}(p) = \tilde{f}(\pi(p)) + |r(x)|^2$$

where $r: N \to [0, \infty): x \mapsto$ distance x to Σ

use some metric

now extend $\hat{f}$ to a smooth function on M

and approximate $\hat{f}$ by a Morse function f agreeing with $\hat{f}$ on N

exercise: for a critical point on Σ show the index of f = index of $\tilde{f}$

Remark: One may use jet transversality to show that for a generic (residual set of) Morse function all the critical points have distinct values

Fact: On a compact manifold M a function $f: M \to \mathbb{R}$ is "stable" $\Leftrightarrow$ f is Morse with critical points having distinct values

a function $f: M \to \mathbb{R}$ is "stable" if "functions near f are "the same" as f up to diffeo"

i.e. $\exists$ open set $U \subset C^\infty(M, \mathbb{R})$ around f s.t.

$\forall g \in U$, $\exists$ diffeos $\phi: M \to M$, $\psi: \mathbb{R} \to \mathbb{R}$ s.t.

$$g = \psi \circ f \circ \phi$$